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Algebraic Thinking: A Case Study on Assessment in Calculus Course

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Abstract. This article presents a case study investigating the role of algebraic thinking in assessment within a calculus course. Algebraic thinking constitutes a fundamental aspect of mathematical proficiency and plays a critical role in the comprehension and application of calculus concepts. The aim of this study is to explore the effectiveness of various assessment strategies in promoting and evaluating students' algebraic reasoning within the context of calculus instruction. The research methodology involved collecting data from a sample of calculus students through multiple assessment instruments, including problem-solving tasks, written examinations, and semi-structured interviews. This study focuses on students' abilities to manipulate algebraic expressions, solve equations, and generalise patterns within calculus problems. The findings from this case study reveal both the strengths and limitations of different assessment methods in capturing students' algebraic thinking skills. The results indicate that traditional assessment formats, such as written exams, tend to evaluate procedural knowledge rather than conceptual understanding and flexibility in applying algebraic techniques within calculus. Conversely, performance-based assessments, such as problem-solving tasks and interviews, offer a richer insight into students' algebraic reasoning. The implications of this study underscore the importance of integrating algebraic thinking into the teaching and assessment practices of calculus. Educators are encouraged to incorporate open-ended problem-solving tasks and reflective assessments that prompt students to apply algebraic methods creatively and analytically. By fostering and evaluating algebraic thinking in calculus, instructors can deepen students' understanding of the subject matter and enhance their overall mathematical proficiency.

INTRODUCTION

In the realm of higher education, particularly within Computer Science degree programmes, mastery of mathematical concepts constitutes a crucial foundation for the development of students' analytical and problem-solving abilities. One of the core modules that plays a significant role in equipping students with these competencies is Calculus. Beyond teaching technical skills for solving equations and mathematical problems, Calculus fosters the development of deeper analytical thinking. A fundamental aspect of understanding and applying calculus concepts lies in algebraic thinking, which involves the ability to manipulate algebraic expressions, solve equations, and generalise mathematical patterns (Katz & Barton, 2007).

Algebraic thinking plays a pivotal role in helping students connect abstract concepts with practical applications in calculus. Previous studies have demonstrated that robust algebraic reasoning can enhance students' overall comprehension of calculus, thereby improving their ability to apply calculus techniques to a range of real-world problems (Stacey & Chick, 2004). Nevertheless, despite the widely acknowledged significance of algebraic reasoning in academic literature, a gap persists in classroom assessment practices. Many of the current assessment methods tend to emphasise procedural knowledge and mechanical skills, while often neglecting the assessment of conceptual understanding and the application of algebraic reasoning in more complex contexts (Tall & Mejía Ramos, 2010).

The primary issue identified in this research is the lack of assessment strategies capable of effectively measuring students' algebraic reasoning within the context of calculus learning. Most assessment methods currently in use tend to measure procedural proficiency rather than students' capacity to grasp calculus concepts deeply and apply them in diverse situations. A widely proposed solution involves the development and implementation of assessment methods that go beyond mechanical skills, encompassing both conceptual understanding and flexibility in the use of algebraic techniques (Black & Wiliam, 2018).

The scholarly literature has identified several specific solutions to improve the assessment of students' algebraic thinking in calculus education. For example, (Jones et all., 2015) advocate for the use of performance-based assessments, such as mathematical problem-solving tasks that require students to articulate their thought processes. This research suggests that by asking students to explain the steps taken in solving problems and to discuss alternative approaches, educators can gain a deeper understanding of their conceptual grasp. This approach facilitates a more comprehensive assessment of algebraic reasoning, as students must demonstrate flexibility in applying algebraic concepts across various calculus scenarios.

Furthermore, Smith (2022) emphasises the importance of reflective assessment, wherein students are asked to identify and explain algebraic patterns they encounter in calculus problems. This approach not only assesses students' understanding of taught concepts but also encourages more critical and analytical thinking. The study found that students engaged in reflective assessment showed significant improvement in their ability to recognise and apply algebraic patterns in more complex contexts, compared to those assessed using traditional methods.

This research aims to explore the effectiveness of various assessment strategies in measuring students' algebraic reasoning. The novelty of this study lies in its holistic approach, which integrates multiple assessment methods, including performance-based and reflective assessments, to gain a more comprehensive picture of students' algebraic reasoning capabilities. The study also seeks to contribute to the existing literature by offering new perspectives on how more comprehensive assessment methods can be integrated into the teaching of calculus (Star, 2021).

The scope of this study includes the analysis of data collected from Computer Science students at Dian Nusantara University who are currently enrolled in a calculus course. The research focuses on students' abilities to solve mathematical problems based on three key indicators: Generational, Transformational, and Meta-Global levels, while also identifying areas of weakness in students' algebraic understanding and skills within the context of calculus.

METHODOLOGY

he research methodology employed in this study adopts a qualitative approach, utilising a case study method to explore and gain an in-depth understanding of assessment strategies for evaluating students' algebraic reasoning in calculus instruction (Creswell, 2014). The research subjects were selected through purposive sampling, comprising Computer Science students at Dian Nusantara University who had completed at least one semester of calculus and exhibited varying levels of mathematical proficiency. From a total of 30 student responses, the response demonstrating the highest level of understanding and algebraic reasoning was selected for detailed analysis. Data collected for this study included written examination results, mathematical problem-solving assignments, and interview transcripts. The research procedure comprised several stages: preparation (literature review and development of research instruments), data collection (written tests, problem-solving tasks, and interviews), data analysis (using thematic analysis techniques to identify patterns and themes related to algebraic reasoning), and reporting of findings. The research instruments were validated by subject matter experts prior to implementation. Data analysis was conducted through the identification of initial codes, grouping of codes into themes, and thematic interpretation to discern patterns within the data. The results from each instrument were cross-compared to construct a comprehensive overview of students' algebraic reasoning abilities. The findings of this study are expected to contribute to the development of more effective assessment strategies in calculus education.

FINDINGS AND DISCUSSION

In the analysis of algebraic reasoning abilities, several indicators are employed to evaluate students' understanding and skills. These indicators include generational, transformational, and meta-global levels of capability. Generational ability refers to students' competence in identifying relevant variables and representing relationships between variables through equations. Transformational ability focuses on students' skills in correctly simplifying and manipulating algebraic forms to arrive at accurate solutions. Meanwhile, the meta-global level assesses students' capacity to use algebra to analyse change, interpret relationships, and predict correct outcomes within mathematical contexts.

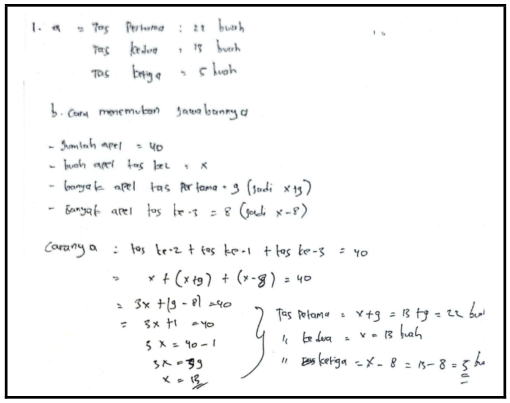

Figure 1 Response of MAM Student Number 1

Response Number 1 in Figure 1 demonstrates a reasonably good understanding in identifying variables and representing the relationships between them within the given problem. The response correctly identifies that the variable x represents the number of apples in the second bag, and subsequently expresses the quantities in the first and third bags as $x + 9$ and $x - 8$, respectively. However, there are errors in the process of simplifying algebraic expressions and in formulating the final equation. For instance, in the final equation-writing step, Response Number 1 fails to correctly construct the equation, leading to inaccuracies in the algebraic form. Although the student ultimately arrives at the correct answer by finding $x = 13$ and accurately calculating the number of apples in each bag, the algebraic manipulation errors suggest that the student (referred to as MAM) still requires a deeper understanding of algebraic transformations. Overall, while Response Number 1 reflects a sound foundational grasp of algebraic concepts—particularly in recognising and representing variables—there are notable shortcomings in the ability to perform algebraic operations accurately. These issues must be addressed to ensure precision in mathematical problem-solving.

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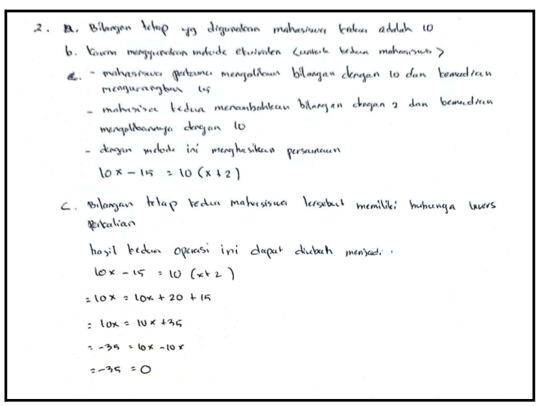

Figure 2 Response of MAM Student Number 2

In terms of the generational indicator, Response Number 2 in Figure 2 demonstrates a sound understanding by identifying the variables and explaining the operations undertaken by the MAM student, as well as successfully representing the problem in the form of the equation $10x - 15 = 10(x + 2)$. However, under the transformational indicator, Response Number 2 encounters difficulty in simplifying and solving the equation. An error occurs when Response Number 2 attempts to simplify the equation and produces an illogical statement, namely $-35 = 0$. This indicates that Response Number 2 has not yet fully mastered the correct application of algebraic operations. Regarding the meta-global level indicator, although Response Number 2 attempts to analyse the relationship between the operations performed by the MAM student by referencing the inverse relationship of multiplication, the error in simplifying the equation renders the analysis inaccurate. Overall, Response Number 2 reflects an adequate basic understanding in recognising and representing variables but requires significant improvement in executing algebraic operations and simplifying equations in order to analyse and solve problems correctly.

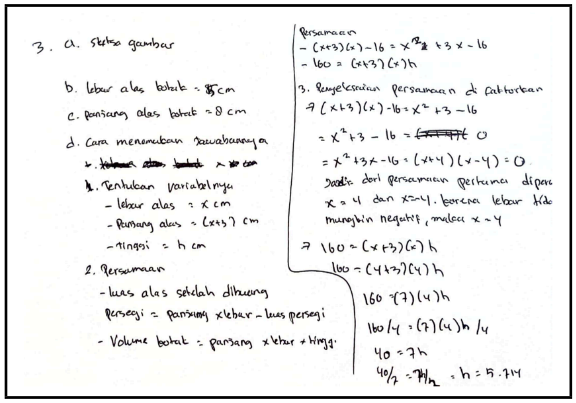

Figure 3 Response of MAM Student Number 3

Response Number 3 in Figure 3 can be analysed based on the indicators of algebraic thinking as follows. Under the generational indicator, Response Number 3 demonstrates strong ability in identifying variables and representing the problem in algebraic form. The MAM student accurately defines the width of the base of the box as x cm, the length of the base as x + 3 cm, and the height as h cm, and attempts to construct an equation that illustrates the relationship among these variables. However, in relation to the transformational indicator, Response Number 3 encounters difficulties in correctly simplifying and manipulating algebraic expressions. Errors emerge in the simplification of the equation, in the factorisation process, and in solving the equation, leading to a result that is irrelevant to the context of the problem. This suggests that Response Number 3 has not yet fully acquired the competence to perform algebraic operations and simplify equations accurately. Regarding the meta-global level indicator, although Response Number 3 makes an effort to use algebra to analyse and predict outcomes, the errors in the earlier steps render the analysis inaccurate. The resulting value, such as h = 5.714, does not align with the problem's context due to flawed analytical processes from the outset. Overall, Response Number 3 demonstrates a good grasp of recognising and representing variables, but further development is needed in simplifying and solving algebraic equations in order to effectively analyse and resolve mathematical problems.

Table 1 below presents an evaluation of several MAM student responses based on these indicators. Each response is analysed to determine the extent to which MAM students are able to meet each indicator and to identify areas in need of improvement. This evaluation aims to provide a clear overview of MAM students' understanding and algebraic skills in the context of mathematical problem solving.

Table 1 Algebraic Thinking Analysis of MAM Students

Nomor	Generasional	Transformasional	Level Meta Global
1	Response Number 1 successfully identifies the relevant variables within the problem. The MAM student assumes that the variable x represents the number of apples in the second bag, and subsequently uses this variable to represent the number of apples in the first and third bags as x + 9 and x − 8, respectively. This indicates that Response	In the process of solving the equation, Response Number 1 attempts to perform algebraic operations to find the solution. However, there are several errors in simplifying algebraic expressions and in the formulation of the equation. For instance, the equation was written as "bag 2 + bag 1 + bag 3 = 40", which should correctly be expressed as *x + (x +*	Despite the errors in certain operational steps, Response Number 1 demonstrates a fundamental ability to use algebra in analysing the relationships between variables and predicting the final outcome. The student made an attempt to evaluate how changes in one variable could influence the overall

	Number 1 meets the first indicator, namely the ability to determine the meaning of variables within a problem. Furthermore, Response Number 1 also successfully represents the problem in terms of the relationships between variables by formulating an equation that involves all three bags, albeit with a minor error in the final equation format.	*9) + (x − 8) = 40*. Although the final answer is correct, the errors in these intermediate steps indicate that Response Number 1 still requires improvement in formulating equivalent algebraic expressions and performing algebraic operations accurately. Response Number 1 ultimately determines $x = 13$ and correctly calculates the number of apples in each bag, yet the algebraic transformations used were not entirely accurate.	result. However, the inaccuracies in the algebraic procedures indicate that Response Number 1's understanding of algebra still requires reinforcement, particularly in ensuring precision when solving mathematical problems.
2	Response Number 2 demonstrates an initial sound understanding in identifying variables and explaining the operations undertaken by the MAM student. The response is able to recognise that both MAM students employed equivalent methods in constructing the equation, reflecting an ability to grasp the meaning of variables within the problem context. Additionally, Response Number 2 successfully represents the relationships among variables by formulating the equation $10x - 15 = 10(x + 2)$. This indicates that Response Number 2 meets the generational indicator, particularly in representing the problem in the form of an equation involving variables.	At this stage, however, Response Number 2 appears to encounter difficulties in simplifying and solving the equation. After formulating $10x - 15 = 10(x + 2)$, the student attempts simplification but makes errors in the process. For example, the expression $10x = 10x + 20 + 15$ is incorrect, as the addition of 15 and 20 should not occur simultaneously on the same side of the equation. This mistake leads to an illogical result, namely $-35 = 0$. Such an outcome indicates that Response Number 2 has not yet fully mastered the transformational indicator, particularly in simplifying algebraic expressions and executing algebraic operations accurately.	Response Number 2 also attempts to analyse the operations by stating that the constants used have an inverse multiplicative relationship. However, this explanation is not supported by accurate algebraic steps in solving the equation. Because the simplification process was flawed, the analysis of inverse multiplication is consequently incorrect. This demonstrates that Response Number 2 is not yet fully capable of using algebra to analyse changes, relationships, and predict accurate outcomes within mathematical contexts.
3	Response Number 3 demonstrates a good initial ability in identifying variables and representing the problem in algebraic form. The response defines the width of the base of the box as x cm, the length of the base as x + 3 cm, and the height as h cm. The MAM student also attempts to construct an equation to describe the relationships among these variables. This indicates that Response Number 3 adequately fulfils the generational indicator, particularly in interpreting the	Under the transformational indicator, Response Number 3 appears to struggle with correctly simplifying and manipulating algebraic expressions. The equation written by the student, such as $(x + 3)(x) - 16 = x^2 + 3x - 16$, contains errors in simplification. Furthermore, additional mistakes are evident when attempting to factorise the equation $x^2 + 3x - 16$, where the factorisation is inaccurate. The student also fails to solve the equation appropriately, as the solution provided, $x = 4$, is not	Although Response Number 3 makes an effort to use algebra to analyse the problem and predict outcomes, the errors in algebraic transformation and equation solving render the analysis inaccurate. For example, the value obtained for h, namely 5.714, lacks contextual relevance due to earlier simplification errors that compromised the entire analytical process. This suggests that Response Number 3 has not yet demonstrated effective use of

	meaning of variables within a problem and in representing the interrelationships among them.	contextually valid. These errors indicate that Response Number 3 has not yet fully developed the skills to simplify and perform algebraic operations correctly, nor to determine an accurate solution to the equation.	algebra to analyse change, examine relationships, and predict correct results within a mathematical context.

Based on the analysis of three MAM students' responses presented in Table 1, students' abilities to solve mathematical problems can be evaluated using three main indicators: Generational, Transformational, and Meta-Global Level. According to Polya (2014),, the process of mathematical problem solving involves systematic steps, starting from understanding the problem, planning a solution, executing the plan, and evaluating the outcome. In this context, the generational indicator reflects the student's ability to comprehend the problem by identifying relevant variables and representing relationships among them. For instance, Response Number 1 shows that the student was able to define the variable x as the number of apples in the second bag and use this to represent the quantities in the other bags. This aligns with the view of Krulik & Reys, (1980), who stated that a sound understanding of variables and equation representation is a crucial initial step in mathematical problem solving.

However, although students demonstrated basic competencies in understanding and representing problems, weaknesses were identified in their transformational skills. Effective algebraic transformation requires the ability to simplify and manipulate expressions correctly—an essential component of mathematical proficiency. As noted by Kieran, (1992), a deep operational understanding is crucial for developing algebraic competence. Responses Number 1 and 2 revealed simplification errors, including inaccurate equation formulation and flawed algebraic logic. For example, Response Number 2 resulted in the illogical equation $-35 = 0$, illustrating a fundamental error in simplification steps. These mistakes suggest that students have not yet fully grasped essential algebraic concepts required for efficient algebraic manipulation (Stacey & MacGregor, 1999)

At the meta-global level, the findings indicate that students' abilities to use algebra as an analytical tool for understanding change, relationships between variables, and predicting outcomes also require further development. Higher-level mathematical understanding, as suggested by Tall & Vinner, (1981), demands the capacity to connect concepts and apply knowledge across contexts. For example, Responses Number 1 and 3 attempted to analyse how a change in one variable influences the overall result, yet operational errors led to inaccurate outcomes. This implies a need for more practice and strengthened conceptual understanding to use algebra effectively in problem analysis across diverse mathematical contexts. Recent research by Stein et all., (2008) also supports the idea that students' skills in algebraic manipulation and analytical thinking can be enhanced through problem-based learning that emphasises both conceptual and procedural understanding.

In terms of assessment, these findings highlight that traditional methods focusing solely on procedural and mechanical skills are insufficient to comprehensively measure students' algebraic thinking. While many students perform well on mechanical tasks, they often struggle with problems requiring conceptual understanding and flexible application of algebraic techniques. Therefore, this study recommends the adoption of more holistic assessment methods, such as performance-based and reflective assessments, which can provide a more accurate picture of students' algebraic thinking and their understanding of calculus concepts (Jones et all., 2015).

In conclusion, although the students' responses demonstrate a sound initial understanding of variables and problem representation, there are significant shortcomings in transformational skills and meta-global application. As Kaput, (1999) noted, achieving mathematical fluency requires students to connect conceptual understanding with procedural skills through consistent practice and varied application. This view is also supported by Rittle-Johnson et all., (2015), who emphasise the importance of integrating conceptual and procedural knowledge to enhance students' mathematical understanding. Thus, strengthening foundational algebraic concepts and providing more opportunities to solve a range of mathematical problems are essential for improving students' abilities to tackle more complex mathematical challenges. A more comprehensive, conceptually grounded assessment approach will help identify weaknesses and provide more effective feedback to support student improvement.

CONCLUSION AND RECOMMENDATIONS

Based on the findings and discussion of this study, it can be concluded that students' algebraic thinking skills in the context of calculus learning still require reinforcement, particularly in the transformational and meta-global aspects. While students generally demonstrate a good initial understanding of fundamental concepts such as variables and problem representation, many struggle with correctly simplifying and manipulating algebraic forms and applying algebra to analyse and solve more complex problems. Errors in equation simplification and algebraic logic suggest that students' conceptual understanding and operational skills remain in need of improvement. Therefore, a more comprehensive approach to both teaching and assessment is required—one that goes beyond procedural skills and includes conceptual understanding and analytical thinking.

To enhance students' algebraic thinking skills, it is recommended that calculus instruction place greater emphasis on conceptual understanding and provide varied practice in algebraic manipulation. The use of performance-based assessments, such as mathematical problem-solving tasks that require students to explain their thinking processes, alongside reflective assessments, in which students identify and describe algebraic patterns within calculus problems, could serve as effective strategies. These forms of assessment not only measure mechanical proficiency but also assess conceptual comprehension and analytical ability, thus offering a more comprehensive view of students' mathematical capabilities. Furthermore, constructive and timely feedback should be provided to help students understand their mistakes and reinforce concepts they have not yet mastered. With these strategies in place, it is expected that students will improve their algebraic thinking skills and be better equipped to apply calculus concepts effectively in real-world contexts.

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